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Quantization of Strongly Interacting Fields

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While perturbative techniques work extremely well for weakly interacting field theories (e.g., QED), they are not useful when studying strongly interacting field theories (e.g., QCD at low energies). In this paper we review Heisenberg's idea about quantizing strongly interacting nonlinear fields, and suggest an approximate method of solving the infinite set of Tamm–Dankoff equations. We then apply this procedure to an infinite-energy, classical flux-tube-like solution of SU(2) Yang–Mills theory and show that this quantization procedure ameliorates some of the bad behavior of the classical solution. We also discuss the possible application of this quantization procedure to a recently proposed strongly interacting phonon model of high-T_c superconductors.

1. INTRODUCTION

In ref. 1 a string model of high-T_c superconductivity was suggested. This model is based on the proposal that phonons have a strong self-interaction. In this case a flux tube filled with phonons appears to form between the Cooper electrons in a manner analogous to QCD, where it is often postulated that the strong interaction of the theory leads to the formation of confining flux tubes between quarks. Just as in QCD, where the formation of such flux tubes leads to a strong binding of the quarks up to a very high temperature when a deconfining phase sets in, so, too, in the strongly interacting phonon picture the formation of such flux tubes is conjectured to raise the temperature at which the Cooper pairing is broken. The possibility of such a strong interaction between phonons in superconductors was also discussed in ref. 2. Such strongly interacting theories can present a challenge in that it is not

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possible to employ standard perturbation theory techniques (i.e., Feynman diagrams) to them. Some time ago Heisenberg conceived of the difficulties in using an expansion in small parameters for strongly interacting quantum field theories as a result of his investigations into the Dirac equation with nonlinear terms (the Heisenberg equation; see, for example, refs. 3 and 4). In these papers he repeatedly underscored that a nonlinear theory with a strong coupling requires the introduction of another quantization procedure. To this end he worked out a quantization method for strong nonlinear fields using the Tamm–Dankoff method. After briefly reviewing Heisenberg's ideas, we apply his quantization method to a classical flux-tube-like solution of the SU(2) Yang–Mills theory to show how this quantization procedure can soften some of the bad behavior of the classical solution.

2. HEISENBERG QUANTIZATION OF STRONGLY INTERACTING FIELDS

Heisenberg's basic idea proceeds from the fact that the n-point Green's functions must be found from some infinite set of differential equations derived from the field equations for the field operators. As an example we present Heisenberg's method of quantization for a spinor field with nonlinear self-interaction.

The basic equation (Heisenberg equation) has the following form:

$$\gamma^\mu \partial_\mu \psi(x) - l^2 \Im[\psi(\bar{\psi}\psi)] = 0 \tag{1}$$

where γ^μ are Dirac matrices; $\psi(x)$ and $\bar{\psi}$ are the spinor field and its adjoint, respectively; $\Im[\psi(\bar{\psi}\psi)] = \psi(\bar{\psi}\psi)$ or $\psi\gamma^5(\bar{\psi}\gamma^5\psi)$ or $\psi\gamma^\mu(\bar{\psi}\gamma_\mu\psi)$ or $\psi\gamma^\mu\gamma^5(\bar{\psi}\gamma_\mu\gamma^5\psi)$. The constant l has units of length, and sets the scale for the strength of the interaction. Heisenberg emphasized that the two-point Green's function $G_2(x_2, x_1)$ in this theory differs strongly from the propagator in a linear theory. This difference lies in its behavior on the light cone: in the nonlinear theory $G_2(x_2, x_1)$ oscillates strongly on the light cone, in contrast to the propagator of the linear theory, which has a δ-like singularity. Heisenberg introduced the τ functions as

$$\tau(x_1 x_2 \ldots | y_1 y_2 \ldots) = \langle 0 | T[\psi(x_1)\psi(x_2)\ldots\psi^*(y_1)\psi^*(y_2)\ldots] | \Phi \rangle \tag{2}$$

where T is the time-ordering operator; $|\Phi\rangle$ is a state for the system described by Eq. (1). Equation (2) allows us to establish a one-to-one correspondence between the system state $|\Phi\rangle$ and the set of functions τ. This state can be defined using the infinite function set of Eq. (2). Applying Heisenberg's equation (1) to (2), we obtain the following infinite system of equations for various τ's:

$$
\begin{aligned}
& l^{-2}\gamma^{\mu}_{(r)} \frac{\partial}{\partial x^{\mu}_{(r)}} \tau(x_1 \ldots x_n | y_1 \ldots y_n) \\
& \quad = \Im[\tau(x_1 \ldots x_n x_r | y_1 \ldots y_n y_r)] \\
& \qquad + \delta(x_r - y_1)\tau(x_1 \ldots x_{r-1} x_{r+1} \ldots x_n | y_2 \ldots y_{r-1} y_{r+1} \ldots y_n) \\
& \qquad + \delta(x_r - y_2)\tau(x_1 \ldots x_{r-1} x_{r+1} \ldots x_n | y_1 y_2 \ldots y_{r-1} y_{r+1} \ldots y_n) + \ldots
\end{aligned} \tag{3}
$$

Equation (3) represents one of an infinite set of coupled equations which relate various orders (given by the index n) of the τ functions to one another. To make some head way toward solving this infinite set of equations, Heisenberg employed the Tamm–Dankoff method, whereby he only considered τ functions up to a certain order. This effectively turned the infinite set of coupled equations into a finite set of coupled equations.

The standard Feynman diagram technique of dealing with field theories via an expansion in terms of a small parameter does not work for strongly coupled nonlinear fields. Heisenberg used the procedure sketched above to study the Dirac equation with a nonlinear coupling. From a more recent perspective it may be interesting to apply the same procedure to nonlinear bosonic field theories such as QCD in the low-energy limit or the recently proposed [1] strongly interacting phonon theory of high-T_c superconductors. In this paper we will apply the Heisenberg method to an infinite-energy, flux-tube-like solution for classical SU(2) Yang–Mills theory. Under certain assumptions we find that the unphysical behavior of the classical SU(2) solution is "smoothed" out when the Heisenberg technique is applied. The formation of flux tubes is an important feature of both QCD (a confining flux tube is thought to form between two quarks) and the strongly interacting phonon model (a flux tube is thought to form between the Cooper electrons, binding them at higher temperatures than is possible in the BCS picture).

3. QUANTIZATION OF SU(2) FLUX TUBE SOLUTION

First we begin by discussing briefly the classical flux-tube-like solution to the SU(2) Yang–Mills theory. The sourceless Yang–Mills equations for SU(2) are given by

$$\nabla_{\mu} F^{a}_{\mu\nu} = 0 \tag{4}$$

where $\nabla_{\mu} = \partial_{\mu} - igA^{a}_{\mu}T^{a}$ is the covariant derivative; T^{a} is an element of the group in some representation; $F^{a}_{\mu\nu} = \partial_{\mu} A^{a}_{\nu} - \partial_{\nu} A^{a}_{\mu} + g\epsilon^{abc} A^{b}_{\mu} A^{c}_{\nu}$ is the SU(2) field strength tensor; A^{a}_{μ} is the SU(2) gauge potential.

To simplify these Yang–Mills equations we make the following cylindrical symmetric ansatz:

$$A_t^1 = f(\rho) \tag{5a}$$

$$A_z^2 = v(\rho) \tag{5b}$$

$$A_\varphi^3 = \rho w(\rho) \tag{5c}$$

Here z, ρ, φ are the standard cylindrical coordinates. Substituting Eqs. (5a)–(5c) into Eq. (4) yields for the Yang–Mills equations

$$f'' + \frac{f'}{\rho} = f(v^2 + w^2) \tag{6a}$$

$$v'' + \frac{v'}{\rho} = v(-f^2 + w^2) \tag{6b}$$

$$w'' + \frac{w'}{\rho} - \frac{w}{\rho^2} = w(-f^2 + v^2) \tag{6c}$$

We further simplify these equations by taking $w = 0$, which yields

$$f'' + \frac{f'}{\rho} = fv^2 \tag{7a}$$

$$v'' + \frac{v'}{\rho} = -vf^2 \tag{7b}$$

These equations can be solved numerically. When this is done it is found that the ansatz function f increases linearly, while v is a strongly oscillating function [5]. The asymptotic behavior of the ansatz functions f, v confirms these numerical calculations:

$$f \approx 2\left[x + \frac{\cos(2x^2 + 2\phi_1)}{16x^3}\right] \tag{8a}$$

$$v \approx \sqrt{2}\,\frac{\sin(x^2 + \phi_1)}{x} \tag{8b}$$

where $x = \rho/\rho_0$ is a dimensionless radius, and ρ_0, ϕ_1 are arc constants. The linearly increasing potential given by the ansatz f is very suggestive of the phenomenological linear confining potentials of QCD. This classical solution has a badly behaved field energy. The energy density for this solution has the following asymptotic proportionality

$$\mathcal{E} \propto f'^2 + v'^2 + f^2v^2 \approx \text{const} \tag{9}$$

where Eqs. (8a) and (8b) have been used. Depending on the initial conditions of the solution the energy density near $\rho = 0$ will be either a hollow (i.e., an energy density less than the asymptotic value) or a hump (i.e., an energy density greater than the asymptotic value). On account of this and the cylindrical symmetry of this solution we call this the "string" solution. The quotation marks indicate that this is a string from an energetic point of view, not from the potential (A^a_μ) or field strength ($F^a_{\mu\nu}$) point of view. The defect in this solution is made apparent when one calculates its total field energy. To do this one must integrate the energy density over all space. Equation (9) implies that this will give an infinite answer. By applying the Heisenberg quantization method to this system, we find that this undesirable behavior of the classical solution is lessened. In order to simplify Heisenberg's quantization method to the present nonlinear equations we make the following assumptions:

1. The degrees of freedom relevant for studying this flux-tube-like solution (both classically and also quantum mechanically) are given entirely by the two ansatz functions f, v of Eqs. (5a)–(5c). No other degrees of freedom arise through the quantization process.

2. From Eq. (8a), f is a smoothly varying function for large x, while v is strongly ocsillating. Thus we take $f(\rho)$ to be almost a classical degree of freedom, while $v(\rho)$ is treated as a fully quantum mechanical degree of freedom. Naively one might think that in this way only the behavior of v would change while f stayed the same. However, since f and v are interrelated due to the nonlinear nature of the equations of motion we find that both functions are modified.

To begin using Heisenberg's quantization scheme for this Yang–Mills system we replace the ansatz functions by operators $\hat{f}(\rho)$, $\hat{v}(\rho)$:

$$\hat{f}'' + \frac{\hat{f}'}{x} = \hat{f}\hat{v}^2 \tag{10a}$$

$$\hat{v}'' + \frac{\hat{v}'}{x} = -\hat{v}\hat{f}^2 \tag{10b}$$

Here the prime denotes a derivative with respect to x. Taking into account assumption (2), we let $\hat{f} \to f$ become just a classical function again, and replace $\hat{v}^2$ in Eq. (10a) by its expectation value:

$$\hat{f}'' + \frac{f'}{x} = f\langle v^2\rangle \tag{11a}$$

$$\hat{v}'' + \frac{\hat{v}'}{x} = -\hat{v}f^2 \tag{11b}$$

Now if we take the expectation value of Eq. (11b) and ignore the coupling to f on the right-hand side, we would have an equation for determining $\langle v\rangle = \langle 0|\hat{v}|0\rangle$. However, the two nonlinear terms on the right-hand side of Eqs.

(11a) and (11b) show that a new object, $\langle \hat{v}^2 \rangle$, enters the picture, so that Eqs. (11a) and (11b) are not closed. To obtain an equation for $\langle v^2 \rangle$ we act on $\hat{v}^2(x)$ with the operator

$$\left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} \right)$$

This gives

$$\hat{v}^{2\prime\prime} + \frac{1}{x}\hat{v}^{2\prime} = -2\hat{v}^2 f^2 + 2\hat{v}^{\prime 2} \tag{12}$$

Taking the expectation value of this equation gives the desired equation for $\langle v^2 \rangle$,

$$\langle v^2 \rangle'' + \frac{1}{x}\langle v^2 \rangle' = -2\langle v^2 \rangle f^2 + 2\langle v'^2 \rangle \tag{13}$$

Again this equation is not closed due to the $\langle v'^2 \rangle$ term. We could again try to find an equation for $\langle v'^2 \rangle$ by the same procedure we employed for $\langle v^2 \rangle$. This equation would also not be closed. Continuing in this way, we would find an infinite set of equations. In order to have some hope of handling this problem we need to make some approximation to cut this process off at some finite set of equations. We try two different approximations for the $\langle v'^2 \rangle$ term and show that both yield similiar large-x behavior that fixes the infinite field energy of the classical solution. First we assume that $\langle v'^2 \rangle \approx \alpha \langle v^2 \rangle$ where α is some constant. This assumption yields the following closed equation set:

$$\langle v^2 \rangle'' + \frac{1}{x}\langle v^2 \rangle' = \langle v^2 \rangle (1 - f^2) \tag{14a}$$

$$f'' + \frac{1}{x} f' = f\langle v^2 \rangle \tag{14b}$$

where we have rescaled the functions as $\alpha^2 x^2 \to x^2$, $\langle v^2 \rangle/\alpha \to \langle v^2 \rangle$, $f/\alpha \to f$. As $x \to \infty$ the asymptotic form of the solution becomes

$$\langle v^2 \rangle \approx v_0^2 \frac{\exp(-\gamma x)}{\sqrt{x}} \tag{15a}$$

$$f \approx f_\infty + f_0 \frac{\exp(-\gamma x)}{\sqrt{x}} \tag{15b}$$

with

$$f_0 = \frac{f_\infty v_0^2}{2(1 - f_\infty^2)}, \qquad \gamma = \sqrt{2(1 - f_\infty^2)} \tag{16}$$

Thus if $|f_\infty \leq 1$, then $\gamma > 0$ and we find that the quantum effects tend to modify the bad long-distance behavior of both ansatz functions.

Instead of using the assumption $\langle v'^2 \rangle \approx \alpha \langle v^2 \rangle$ to close the equations, we could also have made the assumption that $\langle v'^2 \rangle \approx \pm \langle v^2 \rangle'$. Since $\langle v'^2 \rangle$ is positive definite, one picks the $\pm$ sign, so that the right-hand side of this assumption is also positive definite. Under this assumption the equations become

$$\langle v^2 \rangle'' + \left(\frac{1}{x} \mp 2 \right) \langle v^2 \rangle' = -2 \langle v^2 \rangle f^2 \tag{17a}$$

$$f'' + \frac{1}{x} f' = f \langle v^2 \rangle \tag{17b}$$

The approximate solution of Eqs. (17a) and (17b) again has the same functional form as Eqs. (15a) and (15b), but now

$$f_0 \gamma^2 = f_\infty v_0^2, \qquad \gamma^2 \pm 2\gamma = -2f_\infty^2 \tag{18}$$

The second relationship can be written (using the first relationship) as

$$\gamma = \mp f_\infty \left(f_\infty + \frac{v_0^2}{2f_0} \right) \tag{19}$$

Although $\langle v'^2 \rangle \approx +\langle v^2 \rangle'$ leads to unphysical exponentially growing solutions, the assumption $\langle v'^2 \rangle \approx -\langle v^2 \rangle'$ leads to exponentially decaying solutions. Under this latter assumption and the previous assumption ($\langle v'^2 \rangle \approx \alpha \langle v^2 \rangle$) for cutting off the equations we find that that the quantum mechanical treatment of this nonlinear system modifies the bad features of the classical solution. The asymptotic behavior of v goes from being strongly ocsillating [see Eq. (8b)] to decaying exponentially, while the asymptotic behavior of f goes from being linearly increasing [see Eq. (8a)] to also decaying exponentially. If the asymptotic forms for these ansatz functions are used in the energy density $\mathcal{E}$ of Eq. (9), we find that the field energy is now finite. (To calculate $\mathcal{E}$ we would replace the classical terms v'^2 and $f^2 v^2$ by the appropriate quantum operator and take the expectation value. The $\langle v'^2 \rangle$ would be handled according to the assumption we used for closing the equations.)

4. DISCUSSION

In this work we have applied Heisenberg's ideas about quantizing strongly interacting nonlinear fields, not to fermionic fields as Heisenberg did, but to the nonlinear field equations of an SU(2) Yang–Mills gauge theory. Although the solution [Eqs. (8)] for the classical equations of motion [Eqs. (7)] had some interesting features (the linear confining behavior indicated by the ansatz function f and the flux-tube-like structure of the energy density), these features also give the classical solution the unphysical feature of having an infinite field energy. Using Heisenberg's quantization procedure on this system and making some assumptions in order to cut off the infinite equation set, we found that the quantum effects replaced the bad large-distance classical behavior of f, v with physical reasonable exponentially decaying behavior. This results in the field energy of the solution being finite. Now in the small-x region one can still expect to find the interesting behavior (i.e., the linear increase of f) of the classical solution due to asymptotic freedom [6]. In non-Abelian theories the coupling strength can become small at small distance scales (i.e., small x) so that the classical solution should be increasingly valid as $x \to 0$. Actually in this $x \to 0$ limit one cannot use the asymptotic form of f, v of Eqs. (8), but one must investigate the classical equations numerically. When this is done [5] one again finds that f is approximately linearly increasing even as $x \to 0$. The conclusion is that at short distances one has the interesting features of the classical solution, while at large distances the quantum effects, as taken into account via Heisenberg's method, replace the bad long-distance behavior of the classical solution with a physically reasonable behavior.

This quantization procedure that we have applied to the SU(2) flux tube may also be useful in investigating similiar structures in others theories. In QCD it is suggested that the formation of flux tubes between isolated quarks is the mechanism responsible for confinement. However, due to the large coupling strength of QCD the standard perturbative treatment of the quantum effects of the theory via Feynman diagrams is ruled out. The present method may be useful for investigating the flux tube structures of QCD. Recently, a nonlinear, strongly interacting phonon model of high-T_c superconductors was given [1], where the strong interaction between the phonons was postulated to lead to the formation of a flux tube between the Cooper electrons. This allowed the electrons to remain correlated up to much higher temperatures than is normally found in the BCS model. The quantization technique used here for the SU(2) Yang–Mills theory may be useful in studying this model.

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